$M = S^2 \times S^2$ ~ Does there exist a pos. sectional curvature metric on M?

If P is a plane inside T_pM for some $p \in M$.

Then $K(P)$ = Gauss curvature of surface $\exp(P) \subseteq M$

(K; $B_\varepsilon \subset P$.)

Circumference of a circle of radius R in a surface S

$$= 2\pi R\left(1 - \frac{KR^2}{6} + O(R^3)\right).$$

(K: Gauss curvature at p.)



Posit. Sec. Curvature means

$\forall x \in M$, $\forall P \subseteq_{\text{plane}} T_xM$,

$K(P) > 0$.

Standard metric on $(S^2)_1 \times (S^2)_2$

$\forall X \in T_{(p_1,p_2)}(S^2)_1$, $Y \in T_{(p_1,p_2)}(S^2)_2$

Then $\exp(\text{span}(X,Y)) =$ flat torus.

$K(\text{span}(X,Y)) = 0$.

This metric has nonneg. sect curvature.

But at each $(p_1,p_2) \in S^2 \times S^2$,

There is a 2-parameter family of planes P at each (p_1, p_2) for which $K(P) = 0$.

Different metric on $S^2 \times S^2$
Cheeger-Müller metric.

Ingredients

① $SO(3) = \{3\times 3$ orthog matrices w/ det 1$\}$

$= \{(e_1|e_2|e_3) : \{e_1, e_2, e_3\}$ is an oriented on basis of $\mathbb{R}^3\}$.

$\overset{\text{diffeo}}{=} \mathbb{RP}^3 = S^3/_{\text{antip pts.}}$

Lie group. - acts by isometries on S^2.

$\forall\, x \in S^2$, there is a 1-parameter family of $SO(3)$ matrices that fixes x.

(called the isotropy subgroup at x.)

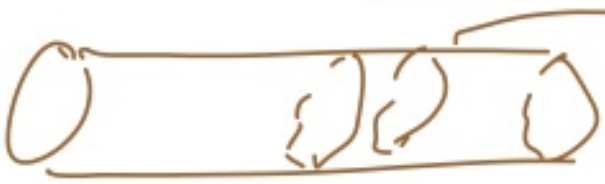
② Let's have $SO(3)$ act by

$A(p_1, p_2) = (Ap_1, Ap_2)$ for $A \in SO(3)$, $(p_1, p_2) \in S^2 \times S^2$.

Isotropy at $(p_1, p_2) = \begin{cases} \{\pm I\} & \text{if } p_1 \neq \pm p_2 \\ \cong SO(2) & \text{if } p_1 = p_2. \end{cases}$

③ Idea of make a flat metric

pos. sec curv:



change metrics on circles (orbits of group action)



Now pos. sectional curvature.

Much trickier in higher dimensions.

④ Construction of C-M metric:

Consider $M \times SO(3)$, with metric $g_0 + \frac{1}{t} Q$

$S^2 \times S^2$ ↗

↗ we can take $t = 1$

where g_0 = standard metric on M,

Q = bimvariant metric on $SO(3)$.

Let π: $M \times SO(3) \to M$

$((p_1, p_2), A) \mapsto A^{-1}(p_1, p_2) = (A^{-1} p_1, A^{-1} p_2)$

fibers of this map

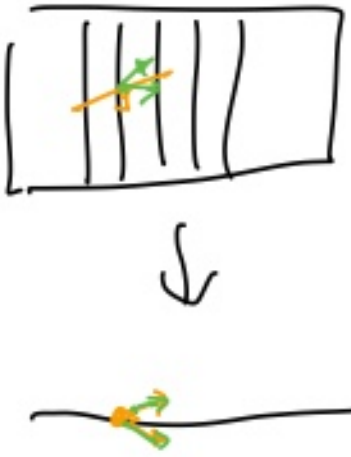
$\pi^{-1}(p_1, p_2) \cong SO(3)$ if $p_1 \neq p_2$

$\pi^{-1}(p_1,p_2) \cong SO(3)$, $p_1 \neq p_2$

Looking at the fibers — they form a Riemannian foliation of $M \times SO(3)$ by fibers — a Riemannian submersion.

The induced metric on M is called the C-M metric.

(ie the transverse metric on $M \times SO(3)$)



Really — the metric on $M = S^2 \times S^2$ is only changing along the fibers — shrinking as you closer to points where $p_1 = p_2$

Formula for metric

$$g = 2\,d\theta^2 + \frac{2\sin^2\theta}{1+2\sin^2\theta}\sigma_1^2 + \frac{2\cos^2\theta}{1+2\cos^2\theta}\sigma_2^2 + \frac{2}{3}\sigma_3^2$$

where. $\sigma_1, \sigma_2, \sigma_3$ are dual to an orn basis e_1, e_2, e_3 of $\mathbb{R}^3$ — translated by $SO(3)$.

points as $S^2 \times S^2$

$p_1 = \cos\theta e_1 - \sin\theta e_2$, $p_2 = \cos\theta e_1 + \sin\theta e_2$, $0 < \theta < \frac{\pi}{2}$

for $p_1 \neq p_2$

As a result

At any $(p_1, p_2) \in S^2 \times S^2$

with the C-M metric,

the planes such that $K = 0$ for them

$= \text{span}\{(b,0),(0,b)\}$

where b is a vector perpendicular to both p_1 & p_2 —

If $p_1 \neq p_2$, there is only one such plane.

If $p_1 = p_2$, then there a circle's worth of such planes.

Brendel & Hung: tweaked C-M metric.

$$g(t) = g_{CM} + t h_1 + t^2 h_2 + t^3 h_3 + \mathcal{O}(t^4)$$

variation of metrics

h_1, h_2, h_3: symmetric tensors.

zero K planes

$F(t) = K(g(t)) \Rightarrow$

$F'(t) = 0 \,|_{t=0}$

$F''(t) = 0 \,|_{t=0}$

for certain $h_3 \Rightarrow F'''(t) > 0 \,|_{t=0}$.